$\ddot{g}$-HIGHER SEPARATION AXIOMS IN IDEAL TOPOLOGICAL SPACES

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ABSTRACT
This paper is aim to study about $\ddot{g}$-higher separation axioms in ideal topological spaces. In particular, it exhibit the properties of $I_{\ddot{g}}$-T_3 space, $I_{\ddot{g}}$-T_4 space and $I_{\ddot{g}}$-T_5 space.

Keywords - *$I_{\ddot{g}}$-T_3 space, $I_{\ddot{g}}$-T_4 space and $I_{\ddot{g}}$-T_5 space.*

I. INTRODUCTION

Separation axioms constitute a classical topic in general topology. These axioms are statements about richness of topology. The equivalence of "Trennungsaxiomen" (German language) is "Separation Axioms". The Separation axioms testify that there are enough open sets to separate the points, closed sets and separated sets. The author [11] have already introduced $I_{\ddot{g}}$-T_0, $I_{\ddot{g}}$-T_1, $I_{\ddot{g}}$-T_2 axioms of separation by imposing certain conditions for separation of points by open sets. Normality was introduced by Vietoris in the year 1921. In this paper we obtain another set of axioms by considering separation of closed sets using open sets and we introduce separation axiom which is stronger than regularity, namely normality.

II. PRELIMINARIES

An ideal I on a topological space (X,τ) is a non empty collection of subsets of X which satisfies (i) $A \in I$ and $B \subset A \Rightarrow B \in I$ and (ii) $A \in I$and $B \in I \Rightarrow A \cup B \in I$. An ideal topological space (X,τ) with an ideal I on X and it is denoted by (X,τ,I). For a subset $A \subset X$, $A^*(I) = \{x \in X: U \cap A \notin I \text{ for every neighbourhood } U \text{ of } x\}$ is called the local function of A with respect to I and τ[3]. We simply write A^* instead of A^*(I) to be brief. For every ideal topological space (X,τ,I), there exists a topology $\tau^*(I)$, finer than τ, generated by $\beta(I,\tau) = \{U - i: U \in \tau \text{ and } i \in I\}$, but in general $\beta(I,\tau)$ is not always a topology[3]. Additionally, $cl^*(A) = A \cup A^*$ defines a kuratowski closure operator for $\tau^*(I)$. A subset A of an ideal space (X,τ,I) is *-closed[1] (resp. *-dense in itselt[2]) if $A^* \subseteq A$(resp.$A \subseteq A^*$).

The family of all α-open sets in(X,τ), denoted by τ^α, is a topology on X finer than τ. The closure of A in (X,τ^α) is denoted by $cl_\alpha(A)$.

Definition 2.1: An ideal I is said to be
1. Ccodense[2] or τ-boundry if $\tau \cap I = \{\emptyset\}$,
2. Completely codense [2] if $PO(X) \cap I = \{\emptyset\}$, where PO(X) is the family of all pre-open sets in (X, τ).

Definition 2.2: A subset A of an ideal topological space(X,τ,I) is said to be

(1) $I_{\ddot{g}}$-closed [13] if $A^* \subseteq U$ whenever $A \subseteq U$ and U is sg-open.
(2) $I_{\ddot{g}}$-open [13] if its complement is $I_{\ddot{g}}$- closed.

Results 2.3:
1. Every open set is $I_{\ddot{g}}$-open.[13]
2. Every T_1-space is $I_{\ddot{g}}$-T_1 space.[11]

Lemma 2.4: Let (X,τ,I) be an ideal topological space. If I is completely codense, then $\tau^* \subseteq \tau^\alpha$.[2]

Lemma 2.5: If (X,τ,I) is any ideal topological space and $A \subseteq X$, then the following are equivalent.[13]

(1) A is $I_{\ddot{g}}$ $-$closed.

(2) $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is sg-open in X.

Lemma 2.6: Let (X, τ, I) be an ideal topological space and $A \subseteq X$. Then A is $I_{\ddot{g}}$ −open iff $F \subseteq int^*(A)$ whenever F is sg-closed and $F \subseteq A$.[13]

Lemma 2.7: Let $(X, \tau. I)$ be an ideal topological space. Then every subset of X is $I_{\ddot{g}}$ −closed iff every sg-open is *-closed.[13]

III. $I_{\ddot{g}}$-T_3 SPACE

Definition 3.1:

An ideal topological space (X, τ, I) is said to be an $I_{\ddot{g}}$-regular space if for each pair consisting of a point x and a closed set B not containing x, there exists disjoint $I_{\ddot{g}}$-open sets U and V such that $x \in U$ and $B \subseteq V$.

Definition 3.2:

An ideal topological space (X, τ, I) is said to be an $I_{\ddot{g}}$-T_3 space if it is $I_{\ddot{g}}$-regular space as well as $I_{\ddot{g}}$-T_1 space.

Theorem 3.3: Every regular space is $I_{\ddot{g}}$-regular.

Proof: The proof is obvious. Since every open set is $I_{\ddot{g}}$-open.(Result 2.3-1)

Theorem 3.4: Every subspace of a $I_{\ddot{g}}$-regular space is $I_{\ddot{g}}$-regular.

Proof: Let X be $I_{\ddot{g}}$-regular space and Y be a subset of X.Let $x \in Y$ and V be closed set in Y not containing x. Then $V = V_1 \cap Y$, where V_1 is a closed set in X. Hence $x \in Y \subseteq X \Rightarrow x \in X$ and V_1 is a closed set in X not containing x. Which implies that there exists U and W in X such that $x \in U$ and $V_1 \subset W$.

$\Rightarrow x \in U \cap Y$ and $V = V_1 \cap Y \subset W \cap Y$.

Therefore Y is $I_{\ddot{g}}$-regular.

Theorem 3.5: (Hereditary properties)

Every subspace of $I_{\ddot{g}}$-T_3 space is $I_{\ddot{g}}$-T_3 space.

Proof: Let X be $I_{\ddot{g}}$-T_3 space and Y be a subspace of X. Now X is $I_{\ddot{g}}$-regular as well as $I_{\ddot{g}}$-T_1 space. Both $I_{\ddot{g}}$-regular and $I_{\ddot{g}}$-T_1 axioms are hereditary properties.

It follows that Y is a $I_{\ddot{g}}$-regular as well as $I_{\ddot{g}}$-T_1 space. Hence Y is $I_{\ddot{g}}$-T_3 space.

Theorem 3.6: Every T_3 space is $I_{\ddot{g}}$-T_3 space.

Proof: Since X is T_3-space it both regular and T_1 space. Every regular space is $I_{\ddot{g}}$-regular (theorem 3.3) and every T_1-space is $I_{\ddot{g}}$-T_1 space.(Result 2.3-2)

Hence X is both $I_{\ddot{g}}$-regular and $I_{\ddot{g}}$-T_1 space.

Therefore X is $I_{\ddot{g}}$-T_3 space.

Theorem 3.7: In an ideal topological space (X, τ, I), the following are equivalent

1. X is $I_{\ddot{g}}$-regular.
2. For every open set V containing $x \in X$, there exists an $I_{\ddot{g}}$-open set U of X such that $x \in U \subseteq cl^*(U) \subseteq V$.

Proof:

$(1) \Rightarrow (2)$

Let V be open subset such that $x \in V$. Then $X - V$ is a closed set not containing x. Therefore there exists disjoint $I_{\ddot{g}}$-open set U and W such that $x \in U$ and $X - V \subseteq W$.

Now $X - V \subseteq int^*(W)$ by lemma (2.6) implies $X - int^*(W) \subseteq V$.

Again $U \cap W = \emptyset$ implies $U \cap int^*(W) = \emptyset$.

Which implies that $cl^*(U) \subseteq X - int^*(W) \subseteq V$.

Therefore $x \in U \subseteq cl^*(U) \subseteq V$.

$(2) \Rightarrow (1)$

Let B be a closed set not containing x. By hypothesis, there exists an $I_{\ddot{g}}$-open set U such that $x \in U \subseteq cl^*(U) \subseteq X - B$. If $W = X - cl^*(U)$, then U and W are disjoint $I_{\ddot{g}}$-open sets such that $x \in U$ and $B \subseteq W$.

Theorem 3.8: If (X, τ, I) is an $I_{\ddot{g}}$-regular, T_1-space where I is completely codense, then X is regular.

Proof: Let B be a closed set not containing $x \in X$. By previous theorem, there exists an $I_{\ddot{g}}$-open set U of X such that $x \in U \subseteq cl^*(U) \subseteq X - B$.

Since X is T_1-space, $\{x\}$ is sg-closed and so $\{x\} \subseteq int^*(U)$(by lemma 2.6). Since I is completely codense, $\tau^* \subseteq \tau^\alpha$ and so $int^*(U)$ and $X - cl^*(U)$ are α-open sets.

Now $x \in int^*(U) \subseteq int\Big(cl\left(int\big(int^*(U)\big)\right)\Big) = G$

and

$B \subseteq X - cl^*(U) \subseteq int\left(cl\left(int\left(int^*(X - cl^*(U))\right)\right)\right) = H$

Then G and H are disjoint open set containing x and B respectively. Therefore X is regular.

Corollary 3.9: If (X,τ) is a T_1-space, $I = \{\emptyset\}$ then the following are equivalent.

1. X is regular.
2. For every open set V containing $x \in X$, there exists a $\ddot{g}$-open set U of X such that $x \in U \subseteq cl(V) \subseteq V$.

Theorem 3.10: If every sg-open subset of an ideal topological space (X,τ,I) is $*$-closed, then (X,τ,I) is $I_{\ddot{g}}$-regular.

Proof: Suppose every sg-open subset of X is $*$-closed, by lemma 2.7, every subset of X is $I_{\ddot{g}}$-closed. Hence every subset of X is $I_{\ddot{g}}$-open. If B is a closed subset not containing x, then $\{x\}$ and B are the required disjoint $I_{\ddot{g}}$-open sets containing x and B respectively. Therefore X is $I_{\ddot{g}}$-regular.

IV. $I_{\ddot{g}}$-T_4 SPACE

Definition 4.1:

An ideal topological space (X,τ,I) is said to be an $I_{\ddot{g}}$-normal space if for every pair of disjoint closed sets A and B, there exists disjoint $I_{\ddot{g}}$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$ and $U \cap V = \emptyset$.

Definition 4.2:

An ideal topological space (X,τ,I) is said to be an $I_{\ddot{g}}$-T_4 space if it is $I_{\ddot{g}}$-normal space as well as $I_{\ddot{g}}$-T_1.

Theorem 4.3: Let (X,τ,I) be an ideal topological space, where I completely codense. Then the following are equivalent.

(1) X is normal.

(2) For any disjoint closed sets A and B, there exists disjoint $I_{\ddot{g}}$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

(3) For any closed set A and open set V containing A, there exists an $I_{\ddot{g}}$-open set U such that $A \subseteq U \subseteq cl^*(U) \subseteq V$.

Proof: $(1) \Rightarrow (2)$ Let X is normal. By definition, for any disjoint closed sets A and B, there exists disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$. Since every open set is $I_{\ddot{g}}$-open (Result 2.3-1).Therefore for any disjoint closed sets A and B, there exists disjoint $I_{\ddot{g}}$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

$(2) \Rightarrow (3)$ Suppose A is closed and V is open set containing A. Since A and $X - V$ are disjoint closed sets, there exists disjoint $I_{\ddot{g}}$-open sets U and W such that $A \subseteq U$ and $X - V \subseteq W$. Since $X - V$ is sg-closed, and W is $I_{\ddot{g}}$-open $X - V \subseteq int^*(W)$.

Then $X - int^*(W) \subseteq V$. Again $U \cap W = \emptyset$

$\Rightarrow U \cap int^*(W) = \emptyset \Rightarrow U \subseteq X - int^*(W)$.

Then $cl^*(U) \subseteq X - int^*(W) \subseteq V$.

Thus U is the required $I_{\ddot{g}}$-open sets with $A \subseteq U \subseteq cl^*(U) \subseteq V$.

$(3) \Rightarrow (1)$ Let A and B be two disjoint closed subsets of X. Then A is a closed set and $X - B$ is an open set containing A. By hypothesis, there exists an $I_{\ddot{g}}$-open set U such that $A \subseteq U \subseteq cl^*(U) \subseteq X - B$. We have theorem, "Let X be an ideal topological space and $A \subseteq X$. Then A is $I_{\ddot{g}}$-open if and only if $F \subseteq int^*(A)$ whenever F is sg-closed and $F \subseteq A$". Therefore $A \subseteq int^*(U)$. Since I is completely codense, By lemma (2.4) $\tau^* \subseteq \tau^\alpha$ and so $int^*(U)$ and $X - cl^*(U) \in \tau^\alpha$. Hence $A \subseteq int^*(U) \subseteq int\left(cl\left(int\left(int^*(U)\right)\right)\right) = G$

and

$B \subseteq X - cl^*(U) \subseteq int\left(cl\left(int\left(int(X - cl^*(U))\right)\right)\right) = H$

Therefore G and H are the required disjoint open sets containing A and B respectively.

Theorem 4.4: Every normal space is $I_{\ddot{g}}$-normal.

Proof: The proof is obvious. Since every open set is $I_{\ddot{g}}$-open (Result 2.3-1).

Theorem 4.5: Every closed subspace of $I_{\ddot{g}}$-normal space is $I_{\ddot{g}}$-normal.

Proof: Let (X,τ,I) be $I_{\ddot{g}}$-normal space and Y be a closed subspace of X. To prove (Y,τ',I) is $I_{\ddot{g}}$-normal space which the relative topology. Let A' and B' be two closed disjoint subsets of Y. Then we have,

$A' = Y \cap A, B' = Y \cap B$, where A and B are closed sets in X. Now Y is closed and A and B are closed. Hence $Y \cap A$ and $Y \cap B$ are closed which are disjoint subsets of X. Hence A' and B' are closed subsets of X which are disjoint. Since (X, τ, I) is $I_{\ddot{g}}$-normal corresponding to the disjoint closed subsets A^* and B^* of X, there exists $I_{\ddot{g}}$-open subsets G and H such that $A' \subset G, B' \subset H, G \cap H = \emptyset$. Now $A' \subset G, A' \subset Y$ $\Rightarrow A' \subset G \cap Y. B' \subset H, B' \subset Y \Rightarrow B' \subset H \cap Y$.

Also $G \cap H = \emptyset \Rightarrow (Y \cap G) \cap (Y \cap H) = \emptyset$. Also G and H are $I_{\ddot{g}}$-open and hence $Y \cap G$ and $Y \cap H$ are $I_{\ddot{g}}$-open in Y. Now corresponds to two closed sets A' and B' of Y, there exists $I_{\ddot{g}}$-open set $Y \cap G$ and $Y \cap H$ such that $A' \subset Y \cap G, B' \subset Y \cap H$ and $(Y \cap G) \cap (Y \cap H) = \emptyset$. Hence (Y, τ', I) is $I_{\ddot{g}}$-normal.

Theorem 4.6: Every subspace of a $I_{\ddot{g}}$-T_4 space is $I_{\ddot{g}}$-T_4.

Proof: Let X be space and Y be subspace of X. Now X is $I_{\ddot{g}}$-normal as well as $I_{\ddot{g}}$-T_1 space. We have both $I_{\ddot{g}}$-normal and $I_{\ddot{g}}$-T_1 space are hereditary properties. If follows that Y is a $I_{\ddot{g}}$-normal as well as $I_{\ddot{g}}$-T_1 space. Hence Y is $I_{\ddot{g}}$-T_4 space.

Theorem 4.7: Every $I_{\ddot{g}}$-T_4 space is $I_{\ddot{g}}$-T_3.

Proof: Let (X, τ, I) be $I_{\ddot{g}}$-T_4. Then it is $I_{\ddot{g}}$-T_1 space and $I_{\ddot{g}}$-normal space. If we prove that $I_{\ddot{g}}$-regular then it will be $I_{\ddot{g}}$-T_3. Let F be any closed subset of X and $x \notin F$. (ie) $x \in X - F$ so that $\{x\}$ is a $I_{\ddot{g}}$-closed set as (X, τ, I) is $I_{\ddot{g}}$-T_1. Consider $\{x\}$ and F since (X, τ, I) is $I_{\ddot{g}}$-normal, there exists two $I_{\ddot{g}}$-open sets G and H such that $\{x\} \subset G, F \subset H, G \cap H = \emptyset$, where $x \in G$ and $F \subset H$ and F is $I_{\ddot{g}}$-closed. Hence (X, τ, I) is $I_{\ddot{g}}$-regular and it being $I_{\ddot{g}}$-T_1. Therefore (X, τ, I) is $I_{\ddot{g}}$-T_3.

Theorem 4.8: Let (X, τ, I) be an ideal topological space. Then the following equivalent

1. X is $I_{\ddot{g}}$-normal.
2. For every closed set A and an open set V containing A, there exists an $I_{\ddot{g}}$-open set U such that $A \subseteq U \subseteq cl^*(U) \subseteq V$.

Proof: $(1) \Rightarrow (2)$ Let A be a closed set and V be an open set containing A. Since A and $X - V$ are disjoint closed sets, there exists $I_{\ddot{g}}$-open sets U and W such that $A \subseteq U$ and $X - V \subseteq W$.

Again $U \cap W = \emptyset. \Rightarrow U \cap int^*(W) = \emptyset$ and so $cl^*(U) \subseteq X - int^*(W)$. Since $X - V$ is sg-closed and W is $I_{\ddot{g}}$-open, $X - V \subseteq W$ implies that $X - W \subseteq int^*(W)$ and $X - int^*(W) \subseteq V$.

Thus we have,

$A \subseteq U \subseteq cl^*(U) \subseteq X - int^*(W) \subseteq V$.

$\Rightarrow A \subseteq U \subseteq cl^*(U) \subseteq V$.

$(2) \Rightarrow (1)$ Let A and B be two disjoint closed subset of X. By hypothesis, there exists an $I_{\ddot{g}}$-open set U such that $A \subseteq U \subseteq cl^*(U) \subseteq X - B$. If $W = X - cl^*(U)$, then U and W are the required disjoint $I_{\ddot{g}}$-open sets containing A and B respectively. Hence (X, τ, I) is $I_{\ddot{g}}$-normal.

Theorem 4.9: Let (X, τ, I) be an ideal topological space, where I is completely codense. If (X, τ, I) is $I_{\ddot{g}}$-normal, then it is a normal space.

Proof: Let X be $I_{\ddot{g}}$-normal. By above theorem 4.8, for every closed set A and an open set V containing A, there exists an $I_{\ddot{g}}$-open set U such that $A \subseteq U \subseteq cl^*(U) \subseteq V$. By theorem 4.3, X is normal.

Theorem 4.10: Let (X, τ, I) be an $I_{\ddot{g}}$-normal space. If F is closed and A is a $\ddot{g}$-closed set such that $A \cap F = \emptyset$, then there exists disjoint $I_{\ddot{g}}$-open sets U and V such that $A \subseteq U$ and $F \subseteq V$.

Proof: Since $A \cap F = \emptyset, A \subseteq X - F$ where $X - F$ is sg-open. $\Rightarrow cl(A) \subseteq X - F$ (Since A is $\ddot{g}$-closed). Since $cl(A) \cap F = \emptyset$ and X is $I_{\ddot{g}}$-normal, there exists $I_{\ddot{g}}$-open sets U and V such that $cl(A) \subseteq U$ and $F \subseteq V$. Thus $A \subseteq U$ and $F \subseteq V$.

Corollary 4.11: Let (X, τ, I) be a $I_{\ddot{g}}$-normal space with $I = \{\emptyset\}$. If F is a closed set and A is a $\ddot{g}$-closed set disjoint from F, then there exists disjoint $\ddot{g}$-open sets U and V such that $A \subseteq U$ and $F \subseteq V$.

Proof: We known that, if (X, τ, I) is an ideal topological space and $I = \{\emptyset\}$. Then A is $I_{\ddot{g}}$-closed if and only if A is $\ddot{g}$-closed. ----(*)

Since $\{\emptyset\}$ is a completely codense ideal, by theorem (4.9), if X is $I_{\ddot{g}}$-normal then it is a normal space. Let F be closed set and A be $\ddot{g}$-closed set disjoint from F. Then there exists disjoint $\ddot{g}$-open sets U and V such that $A \subseteq U$ and $F \subseteq V$.

Theorem 4.12: Let (X, τ, I) be an ideal topological space which is $I_{\ddot{g}}$-normal. Then the following hold.

1. For every closed set A and every $\ddot{g}$-open set B containing A, there exists an $I_{\ddot{g}}$-open set U such that $A \subseteq int^*(U) \subseteq U \subseteq B$.
2. For every $\ddot{g}$-closed set A and every open set containing A, there exists an $I_{\ddot{g}}$-closed set U such that $A \subseteq U \subseteq cl^*(U) \subseteq B$.

Proof: (1) Let A be a closed set and B be a $\ddot{g}$-open set containing A. Then $A \cap (X-B) = \emptyset$, where A is closed and $X-B$ is $\ddot{g}$-closed. By theorem (5), there exists disjoint $I_{\ddot{g}}$-open sets U and V such that $A \subseteq U$ and $X-B \subseteq V$. Since $U \cap V = \emptyset$, we have $U \subseteq X-V$, by lemma $A \subseteq int^*(U)$. Therefore, $A \subseteq int^*(U) \subseteq U \subseteq X-V \subseteq B$.

(2) Let A be $\ddot{g}$-closed set and B be an open set containing A. Then $X-B$ is a closed set containing in the $\ddot{g}$-open set $X-A$. By (1), there exists an $I_{\ddot{g}}$-open set V such that $X-B \subseteq int^*(V) \subseteq V \subseteq X-A$.

Therefore, $A \subseteq X-V \subseteq cl^*(X-V) \subseteq B$. If $U = X-V$, then $A \subseteq U \subseteq cl^*(U) \subseteq B$ and so U is the required $I_{\ddot{g}}$-closed set.

Corollary 4.13: Let (X,τ,I) be an normal space with $I = \{\emptyset\}$. Then the following hold.

1. For every closed set A and every $\ddot{g}$-open set B containing A, there exists an $\ddot{g}$-open set U such that $A \subseteq int(U) \subseteq U \subseteq B$.
2. For every $\ddot{g}$-closed set A and every open set containing A, there exists an $\ddot{g}$-closed set U such that $A \subseteq U \subseteq cl(U) \subseteq B$.

Theorem 4.14: Normality is invariant under continuous, $I_{\ddot{g}}$-closed map and surjection.

Proof: Let X be $I_{\ddot{g}}$-normal space and let $f:(X,\tau,I) \to (Y,\sigma,J)$ be a continuous , $I_{\ddot{g}}$-closed and surjection. To prove Y is $I_{\ddot{g}}$-normal space. Let F_1 and F_2 be disjoint closed sets in Y. Since f is continuous, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are closed in X and $F_1 \cap F_2 = \emptyset$ which implies that $f^{-1}(F_1) \cap f^{-1}(F_2) = \emptyset$. Now X is $I_{\ddot{g}}$-normal space and $f^{-1}(F_1), f^{-1}(F_2)$ are disjoint closed subsets in X. Hence there exists $I_{\ddot{g}}$-open set U and V such that $f^{-1}(F_1) \subseteq U, f^{-1}(F_2) \subseteq V$ and $U \cap V = \emptyset$. Put $W_1 = Y - f(X-U)$. Since f is $I_{\ddot{g}}$-closed map and $X-U$ is $I_{\ddot{g}}$-closed, $f(X-U)$ is $I_{\ddot{g}}$-closed set in Y. Hence W_1 is $I_{\ddot{g}}$-open in Y.

Also $f^{-1}(F_1) \subset U \Rightarrow X-U \subset X - f^{-1}(F_1)$

$\Rightarrow X-U \subset f^{-1}(Y-F_1)$

$\Rightarrow f(X-U) \subset Y-F_1$

$\Rightarrow F_1 \subset Y - f(X-U) = W_1$

$\Rightarrow F_1 \subset W_1$.

So , $f^{-1}(W_1) = f^{-1}\big(Y - f(X-U)\big)$

$= X - f^{-1}\big(f(X-U)\big)$

$\subset X - (X-C)$

$f^{-1}(W_1) \subset U$.

Thus there exists an $I_{\ddot{g}}$-open set W containing F_1 suth that $f^{-1}(W_1) \subset U$.

Similarly, $f^{-1}(W_2) \subset V$.

$f^{-1}(W_1) \cap f^{-1}(W_2) \subset U \cap V = \emptyset$.

$\Rightarrow f^{-1}(W_1 \cap W_2) = \emptyset$

$\Rightarrow W_1 \cap W_2 = \emptyset$.

Thus there exists an $I_{\ddot{g}}$-open W_1 of F_1 and W_2 of F_2 such that $W_1 \cap W_2 = \emptyset$.

Hence Y is $I_{\ddot{g}}$-normal.

V. $I_{\ddot{g}}$-T_5 SPACE

Definition 5.1:

An ideal topological space (X,τ,I) is said to completely $I_{\ddot{g}}$-normal if for any two separated sets A and B of X, there exists $I_{\ddot{g}}$-open sets G and H such that $A \subset G, B \subset H$ and $G \cap H = \emptyset$.

Definition 5.2:

An ideal topological space (X,τ,I) is said to be $I_{\ddot{g}}$-T_5 space if it is completely $I_{\ddot{g}}$-normal as well as $I_{\ddot{g}}$-T_1.

Theorem 5.3: (i) Every completely $I_{\ddot{g}}$-normal space is $I_{\ddot{g}}$-normal.

(ii) Every $I_{\ddot{g}}$-T_5 space is $I_{\ddot{g}}$-T_4 space.

Proof: (i) Let A and B be two disjoint closed subsets of X. Therefore $A = cl(A)$ and $B = cl(B)$ and $A \cap B = \emptyset$. Which implies $cl(A) \cap B = A \cap B = \emptyset$ and $A \cap cl(B) = A \cap B = \emptyset$. $\Rightarrow A$ and B are separated sets. Since (X,τ,I) is completely $I_{\ddot{g}}$-normal space, there exists $I_{\ddot{g}}$-open set G and H such that $A \subset G, B \subset H$ and $G \cap H = \emptyset$. Hence (X, τ, I) is $I_{\ddot{g}}$-normal.

(ii) Let (X, τ, I) is $I_{\ddot{g}}$-T_5 space. Then (X, τ, I) is completely $I_{\ddot{g}}$-normal as well as $I_{\ddot{g}}$-T_1 space. By (i), Every completely $I_{\ddot{g}}$-normal space is $I_{\ddot{g}}$-normal. Hence (X, τ, I) is $I_{\ddot{g}}$-normal as well as $I_{\ddot{g}}$-T_1 space. Therefore (X, τ, I) is $I_{\ddot{g}}$-T_4 space.

Theorem 5.4: Every subspace of completely $I_{\ddot{g}}$-normal space is completely $I_{\ddot{g}}$-normal space.

Proof: Let (X, τ, I) be completely $I_{\ddot{g}}$-normal space and Y be a subspace of X. To prove (Y, τ', I) is completely $I_{\ddot{g}}$-normal which relative topology. Let A and B are separated sets in Y. Then we have, $cl'(A) \cap B = \emptyset, A \cap cl'(B) = \emptyset$.

Now $cl'(B) = cl(B) \cap Y, cl'(A) = cl(A) \cap Y$.

Then

$\emptyset = A \cap cl'(B) = A \cap (cl(B) \cap Y) = \left(A \cap cl(B)\right) \cap Y = A \cap cl(B)$

Since $A \cap cl(B) \subset A \subset Y$.

Similarly, $\emptyset = cl(A) \cap B$.

Which implies that A and B are separated sets in X and since X is completely $I_{\ddot{g}}$-normal space, there exists $I_{\ddot{g}}$-open sets G and H such that $A \subset G, B \subset H$ and $G \cap H = \emptyset$. Now $A \subset G$ and $A \subset Y$

$\Rightarrow A \subset G \cap Y \Rightarrow A \subset G', A \subset H$ and $A \subset Y$

$\Rightarrow A \subset H \cap Y \Rightarrow A \subset H'$,where G', H' open in Y.

$G' \cap H' = (G \cap Y) \cap (H \cap Y) = (G \cap H) \cap Y = \emptyset$.

Hence Y is completely $I_{\ddot{g}}$-normal space.

VI. CONCLUSION

In this paper, we have redefined and explored higher separation axioms in ideal topological spaces. In addition, we also discussed some important results and relationship of it.

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