(r*g*)*-HOMEOMORPHISM IN TOPOLOGICAL SPACES

N. Meenakumari* and T. Indira**
*,**Associate Professor, PG & Research Department of Mathematics,
Seethalakshmi Ramaswami college, Trichy.
Email. Id: meenamega25@gmail.com*
Email. Id: drtindira.chandru@gmail.com **

ABSTRACT
The aim of this paper is to introduce a new type of homeomorphism namely (r*g*)*-homeomorphism.

Keywords - *(r*g*)*-open set, (r*g*)*-closed set , (r*g*)*-closure and (r*g*)*-interior.*

I. INTRODUCTION

N Levine[6] introduced the concept of generalized closed sets and the class of continuous function using (g-open set)semi open sets. Balachandran[1] et al introduced the concept of generalized continuous map in a topological space. Closed mapping in topology was introduced by Malgham [7]. wg-closed maps and rwg-closed maps were introduced by Nagaveni [11]. M. Karpagadevi [5] and A. Pushpalatha[5] studied rw-closed maps and rw-open maps. The aim of this paper is to introduce (r*g*)-closed maps ,(r*g*)*-open maps and a new type of homeomorphism namely (r*g*)*-homeomorphism.

II. PRELIMINARIES

Definition: 2.1 Let A be any subset of a topological space X

1. A is said to be a (r*g*)*closed set [9] if cl(A) ⊆ U whenever A⊆ U and U is r*g*-open. The complement of (r*g*)*closed set is (r*g*)*open.
2. (r*g*)*closure [9] of A is defined as the intersection of all (r*g*)* closed sets containing A.
3. (r*g*)*-interior of A is defined as the union of all (r*g*)*-open sets contained in A.
4. A is said to be a regular generalized closed set (briefly rg-closed) if cl(A) ⊆U whenever A⊆U and U is regular open in (X,τ).
5. A is said to be a generalized pre-closed set (briefly gpr-closed) if pcl(A) ⊆U whenever A⊆U and U is regular open in (X,τ).
6. A is said to be a rwg closed if cl(int(A)) ⊆U whenever A⊆U and U is regular open in (X,τ).

Definition: 2.2 A function f: (X, τ) → (Y, σ) is called

1. (r*g*)*-continuous [10] if the inverse image of every closed set in (Y,σ). is (r*g*)*-closed in (X, τ).
2. (r*g*)*-irresolute map [10] if $f^{-1}(V)$ is a (r*g*)*-closed set in (X, τ) for every (r*g*)*closed set V of (Y,σ).
3. rg-continuous if $f^{-1}(V)$ is a rg-closed set of (X,τ) for every closed set V of (Y,σ).
4. gpr-continuous if $f^{-1}(V)$ is a gpr-closed set of (X,τ) for every closed set V of (Y,σ).
5. rwg-continuous if $f^{-1}(V)$ is a rwg-closed set of (X,τ) for every closed set V of (Y,σ).

Definition: 2.3 A map f: (X, τ) → (Y, σ) is called

1. a rg- closed map if f(U) is a rg-closed set of (Y, σ) for every closed set U of (X,τ).
2. a gpr-closed map if f(U) is a gpr-closed set of (Y,σ) for every closed set U of (X,τ)
3. a rwg-closed map if f(U) is a rwg-closed set of (Y,σ) for every closed set U of (X,τ).

III. (r*g*)*-CLOSED MAPS AND (r*g*)*-OPEN MAPS.

Definition 3.1:

A map f : (X, τ) → (Y, σ) is said to be a (r*g*)*-closed map if the image of every closed set in (X, τ) is (r*g*)*-closed set in (Y, σ).

Theorem 3.2:

Every closed map is (r*g*)*-closed.

Proof

Let f : (X, τ) → (Y, σ) be a closed map. Let U be closed in (X, τ). Then f(U) is closed in (Y, σ). But every closed set is (r*g*)*-closed. ∴ f(U) is (r*g*)*-closed. Hence f is a (r*g*)*-closed map.

Remark:

The converse need not be true as it can be seen from the following example.

Example 3.3:

Let X = {a, b, c}, τ = {ϕ, X, {a}}

Closed sets of (X, τ) are {ϕ, X, {b, c}}

(r*g*)*-closed sets of (X,τ) are {ϕ, X, {b}, {c}, {a, b}, {b, c}, {a, c}}

Let Y = {a, b, c}, and σ = {Y, ϕ, {c}}

Closed sets of (Y, σ) are ϕ, Y, {a, b}

(r*g*)*-closed sets of (Y, σ) are ϕ, X, {a}, {b}, {a, b}, {b, c}, {a, c}

Let f : (X, τ) → (Y, σ) be defined by f(a) = b, f(b) = a, f(c) = c

Now {b, c} is closed in (X, τ). But f({b, c}) = {a, c} is not closed in (Y, σ).

∴ f is not a closed map. But {a, c} is (r*g*)*-closed and hence it is (r*g*)*-closed map.

Theorem 3.4:

Every (r*g*)*-closed map is rg-closed map.

Proof:

Let f : (X, τ) → (Y, σ) be a (r*g*)*-closed map. Let F be a closed set in (X, τ). Since f is (r*g*)*-closed, f(F) is (r*g*)*-closed set in (Y, σ). Since every (r*g*)*-closed set is rg-closed, f is a rg-closed map.

The converse need not be true.

Example 3.5:

Let X=Y={a,b,c}, τ={ϕ, X,{a}}

Closed sets of (X, τ) are ϕ, X, {b,c}.

σ={ϕ, Y, {a}, {b}, {a,b}}

(r*g*)*-closed sets of (Y, σ) are ϕ, Y, {c},{a,c},{b,c}.

rg –closed sets of (Y, σ) are ϕ, Y, {c},{a,b},{a,c},{b,c}.

Now {b,c} is closed in X. f({b,c})={a,b} which is rg-closed but not (r*g*)* closed.Therefore f is a rg-closed not (r*g*)*-closed map.

Proposition 3.6:

Every (r*g*)*-closed map is gpr-closed map.

Proof: The proof follows from the fact that Every (r*g*)*-closed set is gpr-closed.

The Converse need not be true.

Example 3.7:

Let X=Y={a,b,c}, τ={ϕ, X,{b},{a,b}}

Closed sets of (X, τ) are ϕ, X, {c}, {a,c}.

σ={ϕ, Y, {a,b}}

Closed sets of (Y, σ) are ϕ, Y, {c}.

(r*g*)*-closed sets of (Y, σ) are ϕ, Y, {c},{a,c},{b,c}.

gpr –closed sets of (Y, σ) =p(X)

Let f : (X, τ) → (Y, σ) be defined as f(a)=b; f(b)=c; f(c)=a.

Here {c} is closed set in (X, τ). But f({c})={a} is gpr-closed in (Y, σ) but not (r*g*)*-closed.

Proposition 3.8:

Every (r*g*)*-closed map is rwg-closed map.

Proof: proof follows from the fact that every (r*g*)* is closed set is rwg-closed set.

The converse need not be true.

Example 3.9:

Let X=Y={a,b,c}, τ={ϕ, X,{c},{a,c}}

Closed sets of (X, τ) are ϕ, X, {b}, {a,b}.

σ={ϕ, Y, {a}}

(r*g*)*-closed sets of (Y, σ) are ϕ, Y, {b},{c},{a,b},{a,c},{b,c}.

rwg –closed sets of (Y, σ) are ϕ, Y, {a},{b},{c},{a,b},{a,c},{b,c}.

Let f : (X, τ) → (Y, σ) be defined as f(a)=c, f(b)=a, f(c)=b

Here f({a,b})={a,c} and f({b})={a} are rwg-closed sets in (Y, σ) and hence f is rwg-closed map. But {a} is not (r*g*)*- closed in (Y, σ) and hence f is not (r*g*)*-closed map.

Theorem 3.10:

A function f : (X, τ) → (Y, σ) is (r*g*)*-closed iff (r*g*)*-cl (f(A)) ⊆ f(cl(A)) for every subset A of X.

Proof:

Suppose f : (X, τ) → (Y, σ) is a (r*g*)*-closed map.Let A be a subset of X. Then cl(A) is a closed set in (X, τ). Since f is (r*g*)*-closed f(cl(A)) is a (r*g*)*-closed set containing f(A).

Since, (r*g*)*-cl(f(A)) is the smallest (r*g*)*-closed set containing f(A) we have (r*g*)*-cl(f(A)) ⊆ f(Cl(A))

Conversely let F be a closed set in X.

Therefore cl(F) = F.

TPT f is (r*g*)*-closed.

By our assumption

(r*g*)*-cl(f(F)) ⊆ f(cl(F)) = f(F)

But f(F) ⊂ (r*g*)*-cl (f(F))

⇒ (r*g*)*-cl(F) = f(F)

⇒ f(F) is (r*g*)*-closed

⇒ f is (r*g*)*-closed map.

Theorem 3.11:

A mapping f : (X, τ) → (Y, σ) is (r*g*)*-closed if and only if for each subset B of (Y, σ) and each open set U containing $f^{-1}(B)$, there is an (r*g*)*-open set V of (Y, σ) such that B⊂V and $f^{-1}(V)$⊂U.

Proof:

Assume that f is (r*g*)*-closed. Let B⊂Y and let U be an open set of (X, τ) such that $f^{-1}(B)$⊂U.

Volume 6 Issue 4, April 2018, ISSN No.: 2348 – 8190

Now X-U is a closed set of (X, τ) such that $f^{-1}(B) \subset U$.
Since A is (r*g*)*-closed f(X-U) is (r*g*)*-closed in (Y, σ). Put V=Y-f(X-U) is (r*g*)*-open set in (Y, σ),
$f^{-1}(B) \subset U \Rightarrow B \subset V \Rightarrow f^{-1}(B) \not\subseteq x-U \Rightarrow B \not\subset f(X-U) \Rightarrow B \subset Y-f(X-U)=V$.
$f^{-1}(V)= f^{-1}\{Y-f(X-C)\}= f^{-1}(Y)- f^{-1}(f(X-U))$
$=X- f^{-1}(f(X-U)) \subseteq X-(X-U)=U$
$\Rightarrow f^{-1}(V) \subset U$
Conversely, let us assume that $f^{-1}(V) \subset U$
To prove that f is (r*g*)*-closed map.
Let F be a closed set of (X, τ). To show that f(F) is (r*g*)*-closed in (Y, σ).
Now $f^{-1}(f(F^c)) \subset F^c$ and F^c is open in (X, τ). By our assumption there exists a (r*g*)*-open set V in (Y, σ) such that $f(F^c) \subset V$ and $f^{-1}(V) \subset F^c$ and hence $F \subset (f^{-1}(V))^c$.
Hence $V^c \subset f(F) \subset f((f^{-1}(V))^c) \subset V^c$.
$\Rightarrow f(F)=V^c$. Since V^c is (r*g*)*-closed, f(F) is (r*g*)*-closed in (Y, σ). ⇒f is (r*g*)*-closed.

Proposition 3.12:
Composition of two (r*g*)*-closed maps need not be (r*g*)*-closed.
Let X = {a, b, c}, τ = {ϕ, X, {a}}
Closed sets of (X, τ) are {ϕ, X, {b, c}}
(r*g*)*-closed sets of (X,τ) are {ϕ, X, {b}, {c}, {a, b}, {b, c}, {a, c}}
Let Y = {a, b, c}, and σ = {X, ϕ, {c}}
Closed sets of (Y, σ) are ϕ, X, {a, b}
(r*g*)*-closed sets of (Y, σ) are ϕ, X, {a}, {b}, {a, b}, {b, c}, {a, c}
Let f : (X, τ) → (Y, σ) be defined by f(a) = b, f(b) = a, f(c) = c
Then f is (r*g*)*-closed map.
Let g : (Y, σ) → (Z, η) where η={ϕ,Z, {a},{b},{a,b}}.
Closed sets of (Z, η) are ϕ, Z, {b,c},{a,c},{c}.
(r*g*)*-closed sets of (Z, η) are ϕ, Z, {b,c},{a,c},{c}.
Define g : (Y, σ) → (Z, η) by g(a)=a; g(b)=c; g(c)=b.
Now {a,b} is closed in (Y, σ). g({a,b})={a,c} which is (r*g*)*-closed in (Z, η). Hence g is a (r*g*)*-closed map.
Now g∘f: (X, τ)→(Z, η). {b,c} is closed in (X, τ).
(g∘f)({b,c})=g(f({b,c}))=g({a,c})={a,b}.
Which is not (r*g*)*-closed in (Z, η). Hence (g∘f) is not a (r*g*)*-closed map.
The following theorem gives the condition under which the map (g∘f) is (r*g*)*-closed.

Theorem 3.13:
If f : (X, τ) → (Y, σ) is a closed map and g : (Y, σ) → (Z, η) is a (r*g*)*-closed map then the composition g∘f: (X, τ)→(Z, η) is (r*g*)*-closed map.

Proof:
Let F be any closed set in (X, τ). Since f is a closed map, f(F) is closed in (Y, σ). Since g is a (r*g*)*-closed map, g(f(F)) is (r*g*)*-closed in (Z, η). Hence (g∘f)(F) is (r*g*)*-closed in (Z, η). Therefore (g∘f) is a (r*g*)*-closed map.

Definition 3.14:
A map f : (X, τ) → (Y, σ) is said to be (r*g*)*-open map if the image of every open set in (X, τ) is (r*g*)*-open in(Y, σ) .
(i.e) If for every U open in (X, τ), f(U) is (r*g*)*-open in (Y, σ).

Theorem 3.15:
If f : (X, τ) → (Y, σ) is an open map then it is a (r*g*)*-open map.

Proof
Let U be open in (X, τ). Then f(U) is open in (Y, σ). But every open set is (r*g*)*-open which implies f(U) is (r*g*)*-open. Hence f is (r*g*)*-open map.

Theorem :3.16
1. Every (r*g*)*-open map is rg open map.
2. Every (r*g*)*-open map is rwg-open map.
3. Every (r*g*)*-open map is gpr-open map.

The converses need not be true as seen from the following Examples.

Example:
1. Let X=Y={a,b,c}; τ = {ϕ, X, {a}}; σ = {X, ϕ, {a},{b},{a,b}}
   rg-open sets of (Y, σ) are ϕ, Y, {a}, {b}, {c}, {a,b}.
   (r*g*)*-open sets of (Y, σ) are ϕ, X, {a},{b},{a,b}
   Let f : (X, τ) → (Y, σ) be defined by f(a) = c, f(b) = a, f(c) = b.
   Now f({a})={c} which is rg-open (Y, σ) but not (r*g*)*-open in (Y, σ).Hence f is rg-open but not (r*g*)*-open map.
2. Let X=Y={a,b,c}; τ = {ϕ, X, {a,b}}; σ = {X, ϕ, {a}}
   rwg-open sets (Y, σ) are ϕ, Y, {a}, {b}, {c}, {a,b},{a,c},{b,c}.
   (r*g*)*-open sets of (Y, σ) are ϕ, Y, {a},{b},{c},{a,b},{a,c}
   Let f : (X, τ) → (Y, σ) be defined by f(a) = b, f(b) = c, f(c) = a.
   Now f({a,b})={b,c} which is rwg-open (Y, σ) but not (r*g*)*-open in (Y, σ). Hence f is rwg-open but not (r*g*)*-open map.
3. Let X=Y={a,b,c}; τ = {ϕ, X, {c}}; σ = {X, ϕ, {a,b}}

gpr-open sets of (Y, σ) are ϕ, Y, {a}, {b}, {c}, {a,b},{a,c},{b,c}.
(r*g*)*-open sets of (Y, σ) are ϕ, Y, {a},{b},{a,b}.
Let f : (X, τ) → (Y, σ) be defined by f(a) = b, f(b) = a, f(c) = c.
Now f({c})={c} which is gpr-open (Y, σ) but not (r*g*)*-open in (Y, σ). Hence f is gpr-open but not (r*g*)*- open map.

Theorem 3.17:

A function f : (X, τ) → (Y, σ) is (r*g*)*-open iff f(int(A)) ⊆ (r*g*)*-int(f(A)) for every subset A of X.

Proof

Suppose f : (X, τ) → (Y, σ) is (r*g*)*-open. Now Int(A) is open in (X, τ). Since f is (r*g*)*-open f(int(A)) is a (r*g*)*-open set contained in f(A).
∴ f(int(A)) ⊆ (r*g*)*-int(f(A))
Conversely assume f(int(A)) ⊆ (r*g*)*-int (f(A)).
TPT f is a (r*g*)*-open map.
Let U be open in (X, τ). Then int U = U.
By assumption f(U) ⊆ (r*g*)*-int(f(U))
But (r*g*)*-int (f(U)) ⊆ f(U)
⇒ f(U) = (r*g*)*-int f(U)
⇒ f(U) is (r*g*)*-open in (Y, σ)
⇒ f is (r*g*)*-open map.

IV. (r*g*)*-HOMEOMORPHISM

Definition 4.1:

A bijection map f : (X, τ) → (Y, σ) is called a (r*g*)*-homeomorphism if f is both (r*g*)*-continuous and (r*g*)*-open.

Theorem 4.2:

For every bijective map f : (X, τ) → (Y, σ) the following statements are equivalent.

(i) f^{-1} : (Y, σ) → (X, τ) is (r*g*)*-continuous.
(ii) f is a (r*g*)*-open map.
(iii) f is a (r*g*)*-closed map.

Proof:

(i) ⇒ (ii)

Let U be open in (X, τ). Since f^{-1} is (r*g*)*-continuous $(f^{-1})^{-1}$(U) = f(U) is (r*g*)*-open in (Y, σ) which implies f is (r*g*)*-open map.

(ii) ⇒ (iii)

Let F be a closed set in (X, τ), then F^c is open in (X, τ). Since f is (r*g*)*-open f(F^c) is (r*g*)*-open in (Y, σ). But f(F^c) = Y – f(F) implies f(F) is (r*g*)*-closed. Hence f is a (r*g*)*-closed map.

(iii) ⇒ (i)

Let F be a closed set in (X, τ). Then f(F) is ((r*g*)*-closed in (Y, σ). By our assumption $(f^{-1})^{-1}$(F) = f(F) which is (r*g*)*-closed in (Y, σ) which implies f^{-1} is (r*g*)*-continuous.

Theorem 4.3:

Let f : (X, τ) → (Y, σ) be a bijective and (r*g*)*-continuous map. Then the following statements are equivalent.

(i) f is (r*g*)*-open map.
(ii) f is (r*g*)*-homeomorphism
(iii) f is (r*g*)*-closed map.

Proof

(i) ⇒ (ii)

Follows from the assumption

(ii) ⇒ (iii)

By assumption f is, bijective and (r*g*)*-continuous and a (r*g*)*-homeomorphism and hence it is a (r*g*)*-closed map.

(iii) ⇒ (i)

Proof follows by the assumption and by the previous theorem.

V. CONCLUSION

In this paper we have introduced (r*g*)*-closed maps and (r*g*)*-open maps and studied some properties. Using these maps a new class of homeomorphism namely (r*g*)*-homeomorphism was introduced and some properties were studied.

REFERENCES

[1] Balachandran.K ,Sundaram .P and Maki .H, On generalized continuous maps in topological spaces, *Mem.Fac.Kochi univ.ser.A.Maths.*, 12(1991)5-13.
[2] Biswas N, On Characterizations of semi-continuous functions, *Atti, Accad. Naz. Lincei Rend. Cl.Fis.Mat.Natur.*,48(8)(1970),399-402.
[3] Devi .R ,Maki.H and Balachandran. K, semi Genralized closed maps and gernernalized closed maps, *Mem.Fac.Sci.Kochi Univ.Ser.A.Math.*, 14(1993),41-54
[4] Devi .R, Maki .H and Balachandran .K, Generalized α-closed maps and α generalized closed maps, *Indian.J.Pure.Appl.Math*, 29(1)(1998),37-49.

[5] Karpagadevi. M and Pushpalatha. A, RW-closed maps and RW-open maps in topological spaces, *International journal of computer application technology and research*, vol 2, issue 2, 91-93, 2013.
[6] Levine N, Generalized closed sets in topology, *Rend.Circ.Math.Palermo*, 19(2)(1970),89-96.
[7] Malgham. S.R, Generalized closed maps, *J Karnatak Uni.Sci*., 2782-88, 1982.
[8] Meenakumari .N and Indira .T , r*g* closed sets in topological spaces , *Annals of Pure and Applied Mathematics* vol.6, No. 2,2014,125-132
[9] Meenakumari. N and Indira.T , On(r*g*)* closed sets in topological spaces, *International Journal of Science and Research (IJSR)* ISSN (Online): 2319-7064, Volume 4 Issue 12, December 2015
[10] Meenakumari. N and Indira.T, (r*g*)*-continuous map in Topological spaces, *International journal of Development research*, vol.6, issue 04, 7402-7408, April 2016.
[11] Parimelzhagan.R and Nagaveni.N, Weakly generalized closed maps, *world academy of science, engineering and technology* 26, (2009).

Thesis:

[12] Nagaveni N,Studies *on Generalisation of Homeomorphisms in Topological spaces*, Ph.d Thesis,BharathiyarUniversity,Coimbatore.